Review of Probabilistic Models of the Strength of Composite Materials

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Abstract: The available literature concerning probabilistic models describing the strength of composite materials has been reviewed to highlight the important aspects of this behaviour which will be of interest to the modelling and analysis of a complex system. The success with which these theories have been used to predict experimental results has been discussed. Since the brittle reinforcement phase largely controls the strength of composites, the probabilistic theories used to describe the strength of brittle materials, fibres, and bundles of fibres have been detailed. The use of these theories to predict the strength of composite materials has been considered, along with further developments incorporating the damage accumulation observed in the failure of such materials. Probabilistic theories of the strength of short-fibre composites have been outlined. Emphasis has been placed throughout on straightforward engineering explanations of these theories and how they may be used, rather than providing comprehensive statistical descriptions.

1. Introduction

In a variety of industries, fibre composite materials are now used to fabricate many components, which, in certain cases, may even be critical ones. Aircraft and automobiles are examples of vehicles in which the application of fibre reinforced composite materials has been increasing. Industrial piping has also been made of composite materials in many applications. A variety of pressure vessels are also fabricated from composites.

Therefore, in performing a probabilistic safety assessment of a system, one is often confronted with how to deal with the strength of these type of components so as to account for their contribution to system safety. The problem is relatively well understood for metal components but in the case of composite materials specific problems still exist.

There is now a substantial amount of literature available on probabilistic models that describe the strength of composite materials but its use in reliability formulations has been more limited. Furthermore, most of the developments have emphasised material aspects or statistical formulations.

It is the objective of this literature review to summarise some of that work in order to highlight the important aspects of the behaviour of composite materials which will be of interest to the modelling and analysis of a complex system.

One important feature that is different from metal structures is that while the latter generally exhibit a ductile behaviour, composite materials tend to be brittle. Therefore they can be described by theories based on the weakest link concept as will be described here.

Fibre composite materials with a polymeric matrix have their strength determined basically by the strength of their fibres, which carry mainly axial loads and tend to fail in a brittle manner. The failure of brittle materials can be explained by the theory developed by Weibull [1] several years ago. Furthermore, a composite material is made of a large number of small fibres whose properties, dimensions and initial defects can only be described realistically by a probabilistic formulation.

This work presents a state-of-the art survey of the probabilistic formulations available and a critical evaluation of their potential for practical assessments of the uncertainty in strength predictions for those materials.

In general, the failure of a component depends on the material strength, on the loading and on the structural behaviour, which includes interaction between components, boundary conditions and so on. This work will concentrate only on the first one, dealing only with the parameters that affect the material strength.

The joint effect of the probabilistic description of loading, structural response and material strength is dealt with by the reliability analysis of composite material structures. Although significant work has already been carried out in each field, only the material strength is considered in this review. Also, although the definition of composites encompasses a wide range of materials, only polymeric composites are considered here.

2. Probabilistic Theory of the Strength of Brittle Materials

The statistical theories developed to describe the strength behaviour of brittle materials are often generically classed as 'Weakest Link Theory'. The mathematical basis of such theories is given in Section 2.1, and a brief review of their development and application to brittle materials are given in Section 2.2. The probabilistic theories of the strength of the fibres and bundles of fibres, which constitute the main load bearing phase of many advanced composite materials, are then considered in Sections 2.3 and 2.4 respectively.

2.1 Mathematical formulation

The aim of the probabilistic theory of the strength of brittle materials is the prediction of the probability of failure of a given component of such material under a prescribed loading.

The main assumption of this type of model is that the material can be conceptualised as a series of small volumes, the failure of any one of these leading to the failure of the whole component. If a reliability formulation was adopted to describe this model, one would say that the material could be modelled by a series system in which each component would be the elemental volume constituting the material. Since the system would fail whenever the weakest element fails, it is often denoted as a weakest link system.

The elements are considered to be statistically independent and their strength to be identically distributed. If the probability of failure at a given stress σ of each of the elements is given by $F(\sigma)$ then the probability of failure of a chain of n links is given by,

$$F_n(\sigma) = 1 - \left[1 - F(\sigma)\right]^n \quad (1)$$

In 1939 Weibull [1] proposed a specific form of F(σ) which later became known as the Weibull distribution,

$$F(\sigma) = 1 - \exp\left[-\left(\frac{\sigma - \sigma_u}{\sigma_0}\right)^m\right] \quad (2)$$

where σ_u is the threshold stress below which failure does not occur and σ_0 and m are called the *scale parameter* and the *shape parameter,* respectively.

Considering a volume of material comprised of small elemental volumes, δV, instead of a chain of "links" and integrating over the volume gives:

$$F_V(\sigma) = 1 - \exp\left[-VK_S\left(\frac{\sigma_r - \sigma_u}{\sigma_0}\right)^m\right] \quad (3)$$

where K_s is a factor dependent upon the stress distribution and σ_r is a reference stress at a defined point in the material.

In this "three parameter distribution" often σ_u is taken to be zero, resulting in the two parameter form. Rearranging this form and taking logarithms twice gives:

$$\ln\left[\ln\left(\frac{1}{1 - F_V(\sigma)}\right)\right] = m\ln(\sigma) - m\ln(\sigma_0) + \ln(V) \quad (4)$$

The shape and scale parameter respectively may be estimated graphically from the parameters of a least squares fit of the data on a plot of ln(σ) versus the left hand side of this equation for N replications of an experimental strength test for a given volume of material, V. For the i_{th} value in an ordered set of N strengths, $F_V(\sigma_i)$ can be approximated by,

$$F_V(\sigma_i) = \frac{i}{N+1} \quad (5)$$

which is the expected value of the fraction of the population which fails prior to the ith ordered strength.

Alternatively, the shape parameter may be approximated from the coefficient of variation (C. of V.),

$$COV \approx m^{-0.94} \quad (6)$$

One consequence of adopting the Weibull distribution to describe the strength of composite materials is that their probability of failure will depend on the size of the specimens, a feature that is commonly referred to as the 'size effect'. Consider equation (3) for two cases of differing volumes and stress distributions. Equating the probability of failure of each one gives the ratio of the mean stresses as,

$$\frac{\sigma_2}{\sigma_1} = \left(\frac{K_{s1}V_1}{K_{s2}V_2}\right)^{\frac{1}{m}} \quad (7)$$

This describes an effect of both size and stress distribution on the material strength.

2.2 Strength of Brittle Materials

The impetus for the first studies of statistical strength theory was the observed decrease in material strength with size. The earliest example of this work is often attributed to the work of Pierce [3], published in 1926, on the strengths of cotton yarns, based on Griffith's [4] theory of flaws from 1920. Lieblein [5] pointed out that this is preceded by some 35 years by the study of the strengths of long and short bars by Chaplin [6].

However, the first mention of a material strength size effect dates back 400 years. Hertzberg [7] reproduces a sketch and a passage from the notebook of Leonardo da Vinci detailing an experiment exploring the variation of the tensile strength of iron wires with length. He also says that, "The fact that Leonardo repeated his experiments several times to verify his results reflects his concern with the statistical nature of the problem."

Tucker [8] considered the same concepts as Pierce to study concrete in 1927. Then, in 1939, probably the most important work in the field was published by Weibull [1]. The impact of Weibull's work led to the naming of the most commonly used weakest link theory as 'Weibull theory' [9]. Epstein later recognised the close relationship between weakest link theory and the statistical theory of extreme values. A more general derivation for the Weibull distribution using extreme value theory is given in Harlow, Smith and Taylor [10].

The weakest link theory has been used in the design of ceramics for some time and examples of the literature on this subject are numerous. An early example which discusses the theory in some detail is that by Weil and Daniel [11]. Davies [12] gave an easily interpreted overview of the theories used and the mechanics of how to apply them. The same methods had been applied to the brittle strength of steel in an earlier paper by Davidenkov, Shevandin and Wittmann [13].

The strength size effect of a natural anisotropic fibrous composite, wood, is analogous to that for man-made fibre reinforced plastics. This problem has also been approached using Weibull weakest link theory as early as 1966 by Bohannan [14]. Simple Weibull theory was applied to the decrease in strength with volume by Barret [15] and Madsen and Nielson [16]. This theory was extended to allow for the anisotropic nature of wood using a modified Weibull theory in which the effects of length, width and thickness are considered separately [14, 17, 18].

The analysis of the size effect commonly seen in the strength of concrete is usually fracture mechanics based [19-23]. Studies of this material have shown that more than one crack is required for catastrophic failure It has been observed that, for large volumes, the mean strength is constant, but that its standard deviation varies inversely with the square root of the volume. Failure occurs due to a certain number of cracks for a given volume. Jayatilaka and Trustrum [24] defined a material property in terms of this relationship between the critical number of cracks and the volume.

The failure probability of brittle materials can also be derived from the properties of the flaw size distribution of a material and the tensile stress required to propagate a crack of a given size and orientation relative to the applied load. Jayatilaka and Trustrum [24] established a relationship between the properties of the flaw size distribution and the Weibull modulus, and the applicability of their theory to brittle materials is discussed in [25]. The extension of this statistical approach to brittle fracture to describe biaxially loading systems is described in [26]. Multiaxially loaded composites are considered by Wetherhold and Pipes [27]. Five different approaches that lead to the same Weibull distribution of material strength are given by Kittl and Diaz [28]. An extensive review of the state of the art of the Weibull fracture statistics of a wide range of material types and structures is also given by Kittl and Diaz [29].

2.3 Strength of Brittle Fibres

The fibres which constitute the reinforcement phase of a fibre reinforced plastic (FRP) material are generally brittle in nature, and hence, weakest link theory has been applied to the strength of such fibres. Moreton [30] finds a variation in the strength of individual carbon fibres with gauge length, and uses weakest link theory, assuming normally distributed fibre strengths, to give a 'fair' fit to the experimental data. Herring [31] uses the Weibull theory to describe the behaviour of boron fibres, obtaining a fairly good fit to the data. Bader and Priest [32] use the Weibull theory to describe their experimental data of the strengths of single carbon fibres, fibre impregnated tows and glass / carbon fibre hybrid laminates**.** The strength variability of carbon and glass fibres was found to be approximately described by a simple two parameter Weibull distribution by Manders and Chou [33]. One of the most carefully conducted experimental programme on graphite fibres and on graphite composites was completed by Wu and Chou [34]. Besides extensive statistical modeling, the actual data are included as an appendix. They also used the two parameter Weibull distribution for the graphite fibres.

Both E- and S-glass fibres were investigated by Metcalf and Schmitz [35]. They postulated that there were two distinct populations of flaws governing failure, one controlling failure at short gauge lengths and one controlling failure at long gauge lengths. A simple, single exponent Weibull distribution was found to be inadequate, and at least two different exponents were recommended. A similar approach was taken by Phani [36] to give good predictions of the data of Metcalf and Schmitz. Watson and Smith [37] reanalysed Bader and Priest's data, and statistical tests led them to question whether Weibull theory is appropriate (based on a 99% significance test). However, from a practical point of view, they acknowledge the predictive power of the theory. Padgett, Durham and Mason [38] further analysed Bader and Priest's data using a wider class of Weibull models where the parameters are assumed to be functions of fibre length. The use of two models, a power law and a linear model, gave a reasonable fit to the data, but extrapolation to much shorter or longer lengths was not accurate. A theoretical study of the differing effects of surface and volume distributed flaws on failure strength using a Monte Carlo approach enabled Karbhari and Wilkins [39] to describe differing longitudinal and transversal size effects.

An excellent review and analysis of several types of fibres is provided by Smith [40]. Wagner, both individually [41, 42] and together with Pheonix and Schwartz [43], considered the statistics of the strength of Aramid (specifically Kevlar) fibres, considering the separate effects of surface and volume distributed flaws. Knoff [44] found the simple two parameter Weibull model to be sufficient to describe his Kevlar fibre data. However, the ultra-high strength polyethylene fibres studied by Schwartz, Netravali and Sembach [45] do not follow the classical weakest link behaviour. Penning et al. [46] investigated transversal and longitudinal size effects in similar fibres, and state that statistical strength theory is unable to adequately describe their data.

2.4 Strength of Fibre Bundles

The strength characteristics of strong brittle fibres can be described by the weakest link theory discussed in the previous section. It is now of interest to describe, in probabilistic terms, the strength of a bundle of such fibres. The classic model proposed by Daniels [47] is a major improvement over the simple Weibull theory because the failure mechanism is explicitly considered within an element. A chain of elements is again considered, but in this case the elements are assumed to consist of many fibres, as illustrated in Figure 1.

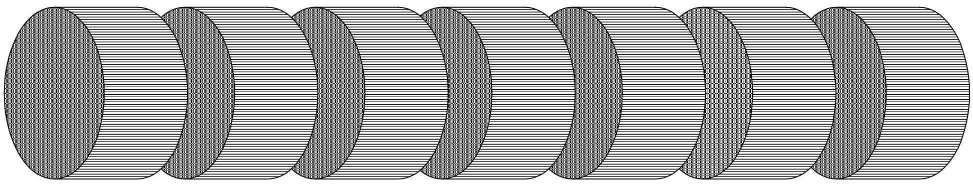

Figure 1: Bundle of Fibres Elements

It is assumed that the applied load is distributed equally among the surviving fibres regardless of the number of failed fibres or their location. The model neglects the interactions between the fibres which arise when the fibres are twisted to form a rope or when they are set in a matrix to form a composite. Daniels' model provides a link between the probabilistic theory of brittle materials, such as glass, carbon or Kevlar fibres, and that of FRP's. Also, bundles of fibres, tows or strands are used as the reinforcement phase in many composites, for example woven-roving E-glass / polyester laminates. Thus, the theory will allow for the prediction of the characteristics of the reinforcing bundles in the composite material.

Daniels' theory allows for any distribution to describe the failure of the individual fibres in a bundle, however, he assumes a Weibull distribution for applications. Further assumptions are that fibre failure is independent of the loading rate and that after failure of one or more fibres, the load is equally redistributed amongst the surviving fibres. he derived an expression for the probability of failure $F^{(N)}(L)$ of a bundle of N fibres subjected to a load L, which is not easily calculated. More importantly, Daniels proved that the bundle strength is asyptotically distributed with a normal distribution. Thus, the bundle distribution for N fibres is approximately a Normal distribution with mean,

$$\mu^* = \max_{x \geq 0} x\left[1 - F(x)\right] \quad \textbf{(8)}$$

and with variance,

$$\sigma = \left(\gamma^* / \sqrt{N}\right)^2 \quad \textbf{(9)}$$

where $$\left(\gamma^*\right)^2 = \left(x^*\right)^2 F\left(x^*\right)\left[1 - F\left(x^*\right)\right] \tag{10}$$

and where x^* is the value at which the mean is defined. For a two parameter Weibull distribution for the fibre strength, the mean and variance can be found explicitly.

McCartney and Smith [48] have proposed a recursion formula which is simpler to compute. Smith [49] proposed an improved relationship which corrected the asymptotic mean;

$$F^{(N)} = \Phi\left(\sqrt{N}\right)\frac{x - 1 - 0.996\beta N^{-2/3}}{\lambda} \tag{11}$$

where β = B/M and Φ is the standard normal distribution.

Barbour [50] then proposed a correction to the asymptotic variance,

$$F^{(N)} = \Phi\left(\sqrt{N}\right)\frac{x - 1 - 0.996\beta N^{-2/3}}{\sqrt{\left(\lambda^2 - 0.317\beta^2 N^{-1/3}\right)}} \tag{12}$$

If the distribution of the strength of the fibre material and the number of fibres are known then these expressions allow a probabilistic description of the strength of an untwisted, loose bundle of fibres.

Phoenix [51] gives a theoretical review of probabilistic models for fibre bundles, including the classical equal load sharing model as a special case of the generalised model where the load on each surviving fibre is not necessarily the same. The model is also extended to describe elastic fibre bundles with random fibre slack, and also includes bundles composed of several types of fibres. This approach was generalised even further for classical fibre bundles to include broad conditions by Harlow and Yukich [52].

The bundles of fibres theory was successfully used by Manders and Chou [33] to assimilate the strengths of loose bundles of carbon fibres with those of single carbon fibres. Weibull parameters were obtained for these carbon fibres and also for E-glass fibres, but the Weibull distribution did not fit the data exactly. Better correlation between the Weibull parameters and the strengths of bundles of up to 6000 carbon filaments was reported by R'Mili, Bouchaour and Merle [53].

The data of Bader and Priest [32] for loose bundles of carbon fibres are reanalysed by Watson and Smith [37]. They observed that, for single fibres and loose bundles of the same length, the theory is remarkably accurate. However, this good agreement is not seen to extend to bundles of differing length. The most likely explanation of this is thought to be some physical interaction between the fibres which would invalidate the theory at longer gauge lengths.

3. Probabilistic Theory of the Strength of Composite Materials

The application of probabilistic fracture theories to the study of composite materials often concerns descriptions of strength 'size effects', whereby the failure stress or strain is observed to decrease with increasing specimen size as described in Section 2.1. Such literature is reviewed in Section 3.1. Weakest link theory describes the failure of brittle materials. However, although most composite materials fail at very low tensile strains, final failure generally occurs after some damage accumulation. The theories developed to accommodate this damage accumulation, and the literature concerning this, are reviewed in Section 3.2. The statistical theories developed to describe the strength of short fibre composites are then considered in Section 3.3.

3.1 Weakest Link Theory

The majority of the work concerning composites has been carried out in the aerospace field using pre-preg carbon-epoxy laminates. A frequently quoted paper is that by Bullock [54] which described a study of two graphite-epoxy systems. Simple Weibull analysis was used to compare the strengths of single strand tows, tensile coupons and flexural three point bending specimens. The effect of volume between the similarly stressed tows and coupons was predicted very accurately by a Weibull distribution that uses both volumes and stresses based on the fibres alone. Similarly, the theory predicts very well the differences in strength observed between the tensile and flexural coupons of equal volume by considering the different stress distributions present. Similar values for the Weibull shape parameter were obtained for tows, tensile coupons and flexural specimens (for 36, 27 and 13 specimens, respectively) for one of the material systems, and an average value of 24 was taken. However, for the other system, more variability in the strengths was observed, and a value of 18 was taken for the shape parameter. Bullock suggested that sufficient specimens must be tested in order to estimate adequately the specific value of the shape parameter for the material considered.

The strength of carbon fibre reinforced plastic (CFRP) was investigated by Hitchon and Phillips [2] who considered two types of both fibre and matrix. The specimens, produced using pultrusion, hand lay-up, filament winding and pre-preg techniques, were most conveniently tested using different test methods. Tensile, hoop-burst and three point flexural tests were carried out. Here, Weibull theory only partially described strength changes; the theory followed tensile and hoop-burst results well, but flexural and tensile strengths were not reconciled. Variations in the shape parameter m (between 10.3 and 38.4) were cited as a possible reason for this; the Weibull analysis being carried out using an average value of 20. Changes in material properties between fabrication routes and failure mode differences between test methods were thought to be responsible for the observed variation in the shape parameter. Also the small number of specimens tested (between 4 and 8) was noted with reference to the confidence in the parameter estimates obtained.

This aspect was further investigated in the second part of the study which sought to resolve these problems by comparing the strengths of different sizes of filament wound hoop-burst specimens. No significant strength decrease ($p < 0.01$) was seen, but it was noted that, for the small number of observations made, the size effects expected for the strength variability seen would be too small to discern. Hitchon and Phillips postulated that Weibull theory may be more applicable when volume is changed through stressed fibre length rather than composite cross-sectional area, and they concluded that further investigation was required.

Tensile and flexural beam column tests of carbon-epoxy were the subject of a series of studies by Kellas and Morton [55], Jackson, Kellas and Morton [56] and Jackson and

Kellas [57]. A size effect was observed for flexural and tensile testing which depended on the lay-up. The applied Weibull theory gave reasonable, but variable, correlation between theory and experiment. No attempt was made to estimate the Weibull shape parameter from the variation of the data, instead this was estimated using equation (7) and the strengths of two specimen sizes. Large variations of m (7.22 to 156 for the tensile tests, and 8.5 to 18.3 for the flexural) were observed across the different lay-ups.

Wisnom [58] observed a size effect for unidirectional carbon epoxy for four-point bending and pinned-end buckling tests. A change in failure mode from tensile to compressive was seen with increasing size. Wisnom postulated that a greater size effect in compression than in tension caused the larger specimens to have lower compressive strengths than tensile strengths. A Weibull shape parameter of 25 was estimated from Wisnom's data using equation (7), but less scatter in the results than this suggests was seen. In order to explain this behaviour, a model of the composite between the extremes of a brittle solid and a loose bundle of fibres was postulated. This model predicted a size effect more dependent upon length than upon the other dimensions.

Hence a second study of specimens of the same cross section, but with varying lengths using three-point bending tests was completed [59]. Here, a Weibull model with both volume and length terms was fitted to the data and found to account for the, lower than expected, variation. Further work [60] found a size effect in the shear strength of carbon epoxy laminates using short beam shear tests. Wisnom draws attention to the caution required when comparing relatively small differences in strength based on observations from a small number of specimen tests.

Grothaus, Hodgkinson and Kocker [61] compared three and four point bending of carbon fibre reinforced plastic using Weibull theory. The stress concentrations at the loading rollers were found to influence failure mode, with steel rollers producing compressive failure and plastic rollers giving tensile failure. By considering tensile and compressive failures separately, Weibull theory was used to explain strength differences.

An investigation into scale effects for fatigue by Chou and Croman [62] also concerns graphite-epoxy. In-line holes drilled in the specimens were used to represent the "links" in weakest link theory. Application of Weibull theory then allowed reasonable predictions to be made. Grimes [63] reviewed selected literature on the static and fatigue scale effects of graphite epoxy bonded and bolted joints.

There is less information available concerning glass reinforced plastic composites, which are of far greater interest to marine engineers. A size effect for the strength of glass fibres was found by Kies [64], and Weibull theory was applied. The behaviour of woven roving glass-polyester was characterised in tension, compression, shear and flexure by Zhou and Davies [65, 66]. Simple Weibull theory was used to explain strength variations with size. The method was found to give reasonable predictions of experimental results, although the lack of statistical analysis of small differences in strength obtained from small numbers of observations does not instil confidence in the conclusions proffered. Wisnom [67] reports a size effect for the interlaminar tensile and shear strengths of unidirectional glass fibre / epoxy measured using curved beam four point bending and short beam shear tests respectively. Again, Weibull theory is used to explain these effects, and Wisnom described the fit as 'reasonable'. Crowther and Starkey [68] found a size effect in the fatigue of unidirectional glass reinforced epoxy and used Weibull statistics to explain this, 'within the limits of its accuracy' and 'in most situations'.

The basis of most of the statistical fracture theories in the literature is the two parameter Weibull distribution but others have also been considered. As described above, Moreton [30] used the normal distribution to describe the strength of carbon fibres. Hwang and Han [69] studied the static strength and the fatigue life distributions of glass fibre reinforced epoxy composites. Normal, log-normal and two parameter Weibull distribution functions were used. They conclude that the Weibull distribution fits the fatigue life data best. For static strength, all three were found to fit the data adequately, but again a Weibull distribution provided the best fit.

The tensile strengths of carbon reinforced and Kevlar reinforced epoxy composites and the compressive strength of glass reinforced epoxy were statistically analysed by Tenn [70]. Using chi-square and Komologorov-Smirnov goodness-of-fit tests and the correlation coefficient from linear regression analysis, normal, log-normal, 2-parameter Weibull and 3-parameter Weibull distributions were evaluated. None of these were rejected at the 5 percent significance level.

Durham and Padgett [71] develop models assuming three parameter versions of the Birnbaum-Saunders distribution and the inverse Gaussian distribution as 'strong competitors to the commonly used Weibull model'. Other distributions used by several authors are also mentioned here. The Birnbaum-Saunders distribution function for failure strengths was also considered by Jerina [72], and was found to apply to graphite epoxy and glass epoxy laminated composites.

3.2 Damage Accumulation Models

One of the assumptions of the theory first put forward by Rosen [73] and Zweben and Rosen [74] is that the constituent fibres of composites do behave as brittle materials, but damage accumulation is also taken into account. The theory is an extension of the bundles of fibres due to Daniels [47]. As described in Section 2.4, Daniels assumed that, for a loose bundle of fibres, when an individual fibre failed load was redistributed equally among the other fibres. However, a fibre reinforced plastic composite material consists of high stiffness and high strength fibres embedded in a relatively low modulus weak polymer matrix. The fibres in a unidirectional, uniaxially loaded composite carry the majority of the load.

In a similar manner to Daniels, Zweben and Rosen hypothesised that when the first of these fibres failed the composite as a whole did not fail, because of load transfer to the surviving fibres. However, they supposed that the load previously supported by the broken fibre is now transferred via the matrix only to the *adjacent* fibres, around the break and then back to the original fibre. The solution of this problem proffered by Zweben and Rosen involves many simplifications and assumptions, but is relatively easily related to the physical system. Hence, the theory of Zweben and Rosen is outlined here, as described in more detail by Batdorf [75, 76]. Later in this section more mathematically correct solutions are described.

A consequence of the shear transfer from broken fibres to intact fibres via the matrix is that, for a certain length on either side of the break, the failed fibre carries less load, whilst those adjacent carry more. This is shown in Figure 2.

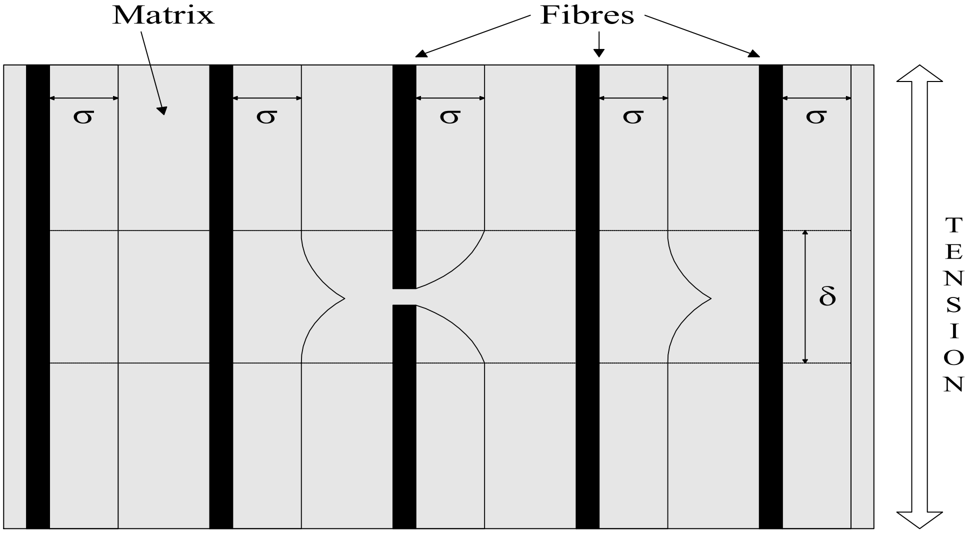

Figure 2: Illustration of Fibre Load Sharing

The length over which this shear transfer occurs is known as the "ineffective length" denoted by δ. Despite the fact that the adjacent fibres now carry more load, since δ is small the probability of a critical flaw occurring in this length is also small, and hence, failure is unlikely. Also the excess load is shared amongst all neighbouring fibres and so the load increase is not large. This type of isolated fibre breakage is termed a "singlet", and as the load is increased more of these will appear. As the load is further increased, it becomes more likely that the over stressed parts of the adjacent fibres should themselves fail. When this occurs there are two adjacent broken fibres and this is termed a "doublet". Still further loading will give rise to more singlets and doublets and then "triplets". This continues until a critical "multiplet" occurs and the process becomes unstable, resulting in the failure of the composite as a whole.

For a fibre, the Weibull distribution may be interpreted as the number of defects unable to sustain a stress σ per unit length of fibre. Hence, considering the two parameter distribution, the number of singlets formed in N fibres of length L is approximately;

$$Q_1(\sigma) = NL\left(\frac{\sigma}{\sigma_0}\right)^m \qquad \textbf{(13)}$$

This assumes that the ineffective length δ is much less than the fibre length L. A further assumption is that N is large so that the number of flaws at the edges of the composite, and hence, not surrounded by other fibres is small. In order to simplify the analysis the ineffective length for a singlet δ_1 is replaced by a conceptual "effective length" λ_1. Each fibre adjacent to a singlet has a maximum increase in stress, σ_{Max}, in the plane of the break. The effective length is that which, when subjected to this maximum stress, has the same probability of failure as the ineffective length subjected to the actual varying stress. The estimated total length of overloaded fibres surrounding the Q_1 singlets is, hence, $Q_1 n_1 \lambda_1$ where n_1 is the number of fibres around each singlet. Defining the ratio of σ_{Max} to σ as C_1 gives the number of failures expected in this length as;

$$Q_2 = Q_1 n_1 \lambda_1 \left(\frac{C_1 \sigma}{\sigma_0} \right)^m \quad (14)$$

This is the number of singlets converted to doublets at stress σ and may be generalised to higher order multiplets;

$$Q_{i+1} = Q_i n_i \lambda_i \left(\frac{C_i \sigma}{\sigma_0} \right)^m \quad (15)$$

It can be seen from equation (15) that a logarithmic plot of Q_i against σ for the ith multiplet will yield a linear graph of slope m_i . This is illustrated for i = 1 to 5 in Figure 3.

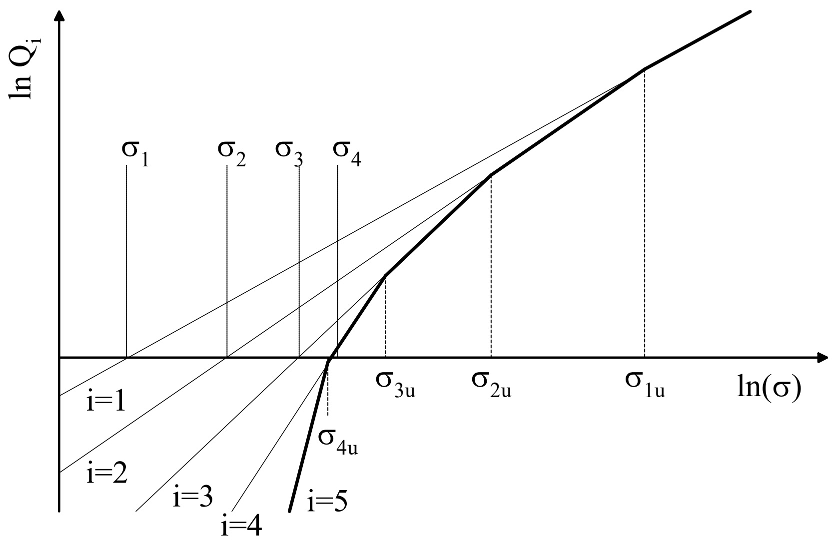

Figure 3: Logarithmic Plot of the Bundle of Fibres Model

As the stress is increased, the first fibre failure occurs at σ_1 where the singlet (i = 1) line cuts the ln Q_i = 0 line. Simple Weibull theory would predict failure of the material as a whole at this point and the i = 1 line does in fact correspond to this theory. However, failure does not occur in this case, and further increase of stress results in more singlets and at σ_2 the first doublets are formed. Since all doublets are formed from singlets there can never be more doublets than singlets. Hence at σ_{1u} the i = 2 line follows that for i = 1. Above this stress any singlets formed are unstable and are immediately converted into doublets. From this it is apparent that failure of the material occurs when the stress at which a certain multiplet first appears is equal or less than that at which it is unstable. This occurs when the envelope of the Q_i lines indicated in bold in Figure 3 cuts the ln(σ) axis. For the case in Figure 3 further increase in stress from σ_2 results in more singlets and doublets until, at σ_3 the first triplets are formed. Further stress produces more singlets, doublets and triplets. At σ_4 the first quadruplet is formed, but this is unstable and failure of the material results.

From equation (15) it is apparent that the expected number of multiplets is proportional to NL, which in a uniform unidirectional composite, is proportional to the composite volume, V. Hence a change in V translates the lines in Figure 3 vertically, changing the values of the intercepts with the ln(σ) axis. This relationship may be represented on a logarithmic plot of failure stress against NL, as shown in Figure 4.

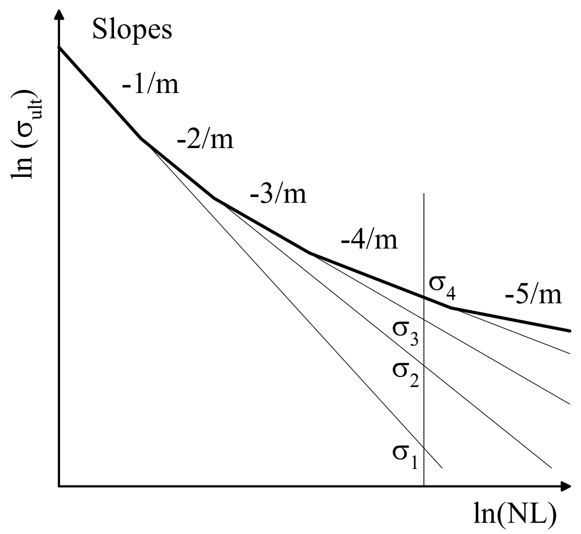

Figure 4: Logarithmic Plot of Bundle of Fibres Model

The failure line is again the bold line, the dashed line indicates the situation shown in Figure 3 where failure occurs when the first quadruplet is formed. As the material volume is increased the failure stress again decreases, i.e. a strength size effect is present. The order of the critical multiplet is seen to increase with composite volume. Also, the dependence of strength on volume decreases as larger amounts of material are considered.

Zweben and Rosen [74] use the first multiple break as the failure criteria in order to compare their theory with experimental data for continuous fibre composites. They reported good correlation with tests on small specimens, but note that, for larger, more damage tolerant composites this may be a conservative estimate of the material strength. Tamuzs [77] used a similar approach, but he used numerical analyses for higher order multiplet failure criteria to give good agreement between theory and experiment.

A significant problem in the use of statistical fracture theories for composite materials is the difficulty of in-situ estimation of parameters such as the ineffective length. Frequently quoted work on the local stress concentrations around broken fibres and the transfer of load to surrounding fibres are those by Hedgepeth [78] and Hedgepeth and Van Dyke [79]. Later work on stress concentrations around broken fibres includes Wolstenholme and Smith [80] and Wolstenholme [81].

Phoenix [51] reviews probabilistic models of fibre/matrix composites with an emphasis on the mathematical assumptions, structure and methods of analysis. The concept of load sharing was further developed by Harlow and Phoenix [82-83], who started with a simplified model of a one dimensional array and obtained ‘cumbersome but *exact* results for small bundles’ assuming the fibre strengths followed the Weibull distribution. Here, it was conjectured that, under the Weibull distribution for the strength of single fibres, a composite specimen has strength which approximately follows another Weibull distribution. Conservative analytical bounds were then obtained [84, 85] for failure based on the occurrence of two or more adjacent broken fibres in the composite. In this paper a recursive technique for generating probabilities for larger bundles from those of smaller bundles was devised. This was further developed [86, 87], by incorporating the occurrence of more than two adjacent failed fibres, and the collapse of the bundle at a critical number of such failures.

An asymptotic approximation to the chain-of-bundles model for fibrous composites was first developed by Smith [49]. Further asymptotic analyses were reviewed by Phoenix and Smith [88], laying the groundwork for the extension of the chain-of-bundles model to cover the three dimensional case. This three dimensional case was then developed by Smith et al. [89] for parallel filaments arranged laterally in a hexagonal array. Barry [90] considers the sixteen nearest fibres to a failed fibre to be subjected to stress concentrations, dividing these into primary and secondary fibres. It is suggested that discrepancies between theoretical and experimental carbon fibre data are due to difficulties in estimating the fibre pull out length which is used to define the 'positively affected length' analogous to the ineffective length as described above.

Batdorf and Ghaffarian [91] found that correlation between Bullock's [54] graphite epoxy data and damage accumulation models was only good when it was assumed that the overloaded length at failed fibre tips was unrealistically large. They postulated that this was due to irregular fibre spacing, and they developed a theory to allow for this, but they found that the fit of this model is unreliable. Smith [92] suggests that the shortcomings of the theories may be that the diameters of the fibre are variable.

Bader and Priest [32] discuss a 'hybrid effect' where the failure stress of a laminate containing both carbon and glass fibres is greater than that of a laminate containing just one type of fibre. They are unable to satisfactorily describe this effect theoretically, as is Smith [92]. Extensions of the probabilistic models of composite strength in order to account for this are proposed by Manders and Bader [93], Bader, Smith and Pitkethly [94], Harlow [95], Fariborz, Yang and Harlow [96] and Fariborz and Harlow [97].

The statistical composite fracture theories for time dependent failure are mathematically developed by Phoenix [51], Phoenix and Kuo [98] and Phoenix, Schwartz and Robinson [99].

Work concerning the statistical theory of single fibres impregnated in resin by Baillie and Bader [100] and by Joffe, Varna and Berglund [101] sought to provide information on the behaviour of fibres in-situ in a resin matrix. Beyerlein and Phoenix [102] move one step closer to real composites by studying the behaviour of microcomposites with four carbon fibres in epoxy resin. Two damage accumulation models which lead to the use of three-parameter generalisations of Birnbaum-Saunders distributions are used by Padgett [103] to describe the strength of composite materials. Both models are seen to give better fits to experimental carbon micro-composite data of Bader and Priest [32] than the more commonly used Weibull distribution based theory.

Much of the literature described above is very mathematical in nature, and is not aimed at the engineer. However, there are several overviews of the field of probabilistic strength theories of composites which are more easily digested by the engineer. Various statistical aspects of the fracture of composites were discussed by Kelly and MacMillan [104], and Argon [105] gives a comprehensive description of the models commonly used. Batdorf [75, 76] provided excellent overviews of both simple weakest link and bundles of fibres models for fibrous composites. He took the view that, although the simplifications made by Zweben & Rosen [74] lead to a non-exact mathematical solution, the increase in accuracy achieved by such an exact solution is small when compared to the errors inherent in estimating the model parameters such as the ineffective length. He also recognised the advantages of a model easily reconcilable with physical quantities over abstract statistical models.

Not all of the literature concerning the statistical fracture of laminated composites attempts to verify the models derived and, as for fibres, conclusive evidence either for or against the models is not gained from the experimental data. The models appear to give adequate qualitative predictions but are quantitatively inaccurate. Model simplifications (such as fibre arrangement, load sharing and diameter assumptions) as well as the difficulties in estimating parameters (for example ineffective length and stress concentration factors) have been suggested as possible reasons for this. Another supposition is that the theory does not allow for the fact that sources of variation, other than those due to flaws, are inevitably present.

Rosen [73] verified qualitatively the progressive and random nature of fibre fractures before final failure of a single layer glass laminate by experimental observation. Zweben and Rosen [74] analysed Rosen's strength data and found that their theory predictions correlated well with the data for small specimens. However, they questioned whether these results could be extrapolated to larger volumes such as those found in structures.

Batdorf and Ghaffarian [91] re-analysed Bullock's [54] data and found that their model was appropriate only when the estimate of the ineffective length parameter was unrealistically large. Bader and Priest were unable to reach any firm conclusions about the agreement with theory for data from single carbon fibres and impregnated bundles. Reanalyses of these data by Smith [92] also failed to assimilate theory and experiment exactly, suggesting that this may be due to variations in fibre diameter.

In order to simplify the problem, Beyerlein and Phoenix [102] considered carbon fibres and simple micro-composites of four fibres in an epoxy matrix. However this approach, together with the inclusion of some measure of the variations in the fibre diameter, still gave only a partial fit to the experimental data. Manders and Chou [33] concluded that it is unrealistic to expect better agreement between theory and practice, in view of the approximations in the models and in the chosen values of parameters such as ineffective length. Good predictions between the strengths of carbon fibres, minitows and tows were achieved by Batdorf [75] but the variation of strengths was much greater than expected. He suggested that this is due to sources of variation other than those due to flaws.

3.3 *Strength of Short-Fibre Composites*

Discontinuous, or short fibre composites have, as the reinforcement phase, fibres which are shorter than those in higher performance laminates. However, the length is still relatively large compared to the diameter. These short fibres, or 'whiskers', may be aligned with one another or randomly orientated. This type of composite materials provide a low-cost alternative to high performance composites when the specific strength requirements are not as critical. Even so, as recognised by Chou and Kelly [106, 107], there is considerable variability in the strength of the material due to the variability in fibre location, orientation and strength.

For short-fibre reinforced composites a theory based on the fracture analysis of the material is found to be more appropriate than those used for long fibre bundles described in the previous two sections [108, 109]. The fracture behaviour of such materials is dominated by fibre fracture and fibre pull-out. Microcracks are likely to occur at the fibre ends and to propagate during loading. When the load bearing capacity of the matrix is lost, the load immediately prior to fracture is borne by the remaining fibres which bridge the critical zone or gap. Fibres that end in the damaged zone are assumed to pull out of the matrix. Those which bridge the gap are assumed to be long enough to fracture rather than to pull out. Hence, the fracture strength of the composite depends, not only upon the fibre strength distribution, but also upon the number of bridging fibres and their orientation and length.

Wetherhold [110] developed a probabilistic theory for the strength of short fibre composites initially presented by Fukuda and Chou [111, 112]. He considered a composite with an aligned system of short bundles of fibres. He assumed that the fracture strength of such a composite where the matrix has been damage in a critical zone is given by the strength of the fibre bundles bridging that zone. Initially the variability in fibre strength is assumed to be small, when compared to the variability in the location of the fibres, and hence, is ignored. The cumulative distribution function for the composite material strength, σ_u , can then be based on the number of bridging fibre bundles. This function is assumed to be of a binomial form. Then, a conditional probability approach is used to include the effect of the variability in the fibre strength. By assuming that the fibre strengths are described by a Weibull distribution, Wetherhold derives an expression for the probability of failure of the composite,

$$P\left[\frac{L}{bc} \leq x\right] = P\left[\Phi \leq \frac{x-\mu}{\bar{\sigma}}\right] \quad (16)$$

where $x = \sigma_f A_f n / bc$ **(17)**

$$A_f = n\pi d_f^2 / 4 \quad (18)$$

$$\mu = n\sigma_0 (lm)^{-1/m} \exp(-1/m) \quad (19)$$

$$\bar{\sigma}^2 = s_\tau^2 \left[n\left\{1-\exp(-1/m)\right\}\exp(-1/m)\right] \quad (20)$$

$$s_\tau^2 = (lm)^{-1/m} \sigma_0 \quad (21)$$

and L is the load on the composite, b is the composite breadth, c is the composite depth, σ_f is the fibre strength, A_f is the fibre cross-sectional area, n is the number of fibres in each bundle, d_f is the fibre diameter, Φ is the standard normal variate, σ_0 is the Weibull scale parameter, l is the fibre length and m is the Weibull shape parameter. It should be noted that the nomenclature used here differs from that used by Wetherhold. Another attempt to model this class of composites is given by Harlow and Phoenix [113].

Wetherhold's model indicates that, for a given fibre volume fraction, the composite strength increases with increasing fibre length. As the fibre length is increased, the effect of the reduction in the number of fibres is more than offset by the greater probability of each fibre bridging the critical damage zone. The coefficient of variation of composite strength decreases with increasing fibre length. Considering the fibre strength to be a random variable, rather than a constant, reduces the composite strength. This is due to the fact that the mean of the Weibull distribution is greater than the median value. In this case this means that the majority of the fibres are of a lower strength. Higher values of the Weibull shape parameter leads to an increase in the mean of the composite strength, but a reduction in its coefficient of variation. In the limiting case of a very large shape parameter, the two cases of variable and constant fibre strengths give identical results.

Wetherhold [114] then went on to consider composite materials with non-aligned fibres by including the effect of the fibre orientation distribution function on the number of fibre bundles bridging the critical damage zone. In a further paper Wetherhold [115] allows for the effects of fibre cross-over. The stress in a longitudinal fibre is decreased in the vicinity of a neighbouring fibre which crosses above or below. A simulation based on the extended theory indicated that the effect of fibre crossover is to increase the composite strength due to this effect. It is possible to have different fibre orientations with the same elastic properties but different crossover densities and, hence, strength. The density of crossovers will depend upon the fibre orientation, length and, in reality, number random variables. The crossover density is roughly proportional to the square of the fibre length, but the coefficient of variation of this density decreases with increasing fibre length. It is noted that comparison with experiments is difficult because the in-situ fibre strength is not easily assessed since the fibres are degraded by the chopping and flow processes of manufacturing. The increased energy absorption during fracture for such composites with very short, poorly bonded fibres is developed using probabilistic principles by the same author [116].

Wetherhold [110] recognises that "One difficulty in making comparisons with data (besides its scarcity) is the determination of *in situ* fiber strength, since the short fibres are invariably degraded by chopping and by the flow processes during moulding.".

4. Concluding Remarks

The literature and background of probabilistic models of the strength of polymeric matrix composite materials have been reviewed. A comprehensive, but not exhaustive, reference list has been provided which enables further research into more specialised, or related areas.

The main theoretical models used have been detailed with the engineer, rather than the statistician, in mind. The way in which these models may be used to analyse experimental data has also been described.

There is a body of literature suggesting that these theories may be used to describe the strength behaviour of composite materials, especially for ‘size effects’. However, the evidence is not wholly consistent. The reason for this appears to be three-fold; a wide range of both materials and test methods are covered, manufacturing variations may mask the trends present, and the *in situ* estimation of the parameters involved is difficult. This is exacerbated by the common problem of a lack of composite material property data.

The theories developed thus far can be statistically fairly advanced, even for the relatively simple geometry’s considered. Developments to the theories to encompass the more realistic non-uniformity of composite materials could be difficult, and will almost certainly lead to even more complex theories.

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